

Some arithmetic properties of matroidal ideals

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Abstract. In this paper, we study various properties of matroidal ideals.

1 Introduction

Let K be a field and $R = K[x_1, \dots, x_n]$ be the polynomial ring in n variables over K with each $\deg x_i = 1$. If $u = x_1^{a_1} \cdots x_n^{a_n}$ is a monomial of R , then we denote the support of u by $\text{supp}(u) = \{x_i \mid a_i \neq 0\}$. For a monomial ideal $I \subseteq R$, $G(I)$ is denoted for the set of its unique minimal monomial generators. We call a monomial ideal I a *matroidal ideal* if each member of $G(I)$ is square-free (i.e., I is reduced) and that the following exchange condition is satisfied: for any $u = x_1^{a_1} \cdots x_n^{a_n}, v = x_1^{b_1} \cdots x_n^{b_n} \in G(I)$, if $a_i > b_i$ for some i , then there exists some j with $a_j < b_j$ such that $x_j u / x_i \in G(I)$ ¹.

In other words, the set $\mathcal{B}(I) = \{\text{supp}(u) \mid u \in G(I)\}$ satisfies the following exchange condition:

- (B) If B_1 and B_2 are elements of $\mathcal{B}(I)$ and $x \in B_1 - B_2$, then there is an element $y \in B_2 - B_1$ such that $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}(I)$.

It follows from [5, Theorem 1.2.3] that there is a “matroid” having $\mathcal{B}(I)$ as its collection of bases (maximal independent sets). Since each maximal independent set of a matroid has the same cardinality (see [5, Lemma 1.2.4]), each monomial $u \in G(I)$ must be of the same degree, say d , and we call this number d the degree of the matroidal ideal I .

The matroid theory is one of the most fascinating research area in combinatorics which has many links to graphs, lattices, codes, and projective geometry. For the interested reader, we refer the textbooks [5] or [7]. In this paper, we focus on some arithmetic properties held by a matroidal ideal.

It is known that a matroidal ideal has linear quotients (cf. [1, Theorem 5.2]). We first discuss the linear quotient index $q(I)$ of a matroidal ideal in section two and get the following result.

Theorem 2.5. *Let I be a matroidal ideal of degree d in the polynomial ring $R = K[x_1, \dots, x_n]$ with $\text{supp}(I) = \{x_1, \dots, x_n\}$. Then $q(I) = n - d$.*

With this result and the fact [3, Corollary 1.6] we obtain that the projective dimension of a matroidal ideal is $\text{pd}_R(I) = n - d$.

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¹ I is called a *polymatroidal ideal* when not requiring the square-free assumption(see[2]).

An ideal is *unmixed* if all its prime divisors are of the same height. It is known that the Cohen-Macaulay ideals hold this property. In section three, we discuss the unmixed matroid ideal and find the relation between the height and the degree of a matroid ideal as follows. (for the “*” product of ideals, please see Definition 2.1)

Theorem 3.6. *Let $I \subseteq K[x_1, \dots, x_n]$ be an unmixed matroidal ideal of degree d with $\text{supp}(I) = \{x_1, \dots, x_n\}$ and $n \geq 2$; then $h + d - 1 \leq n \leq hd$, where h is the height of I . In particular, $n = h + d - 1$ if and only if I is square-free Veronese; and $n = hd$ if and only if $I = J_1 * J_2 * \dots * J_d$, where each J_i is generated by h distinct variables.*

For an ideal I , the minimal number of elements which generate I up to radical is called the *arithmetical rank* of this ideal and is denoted by $\text{ara } I$. When this numerical invariant equals to the height of I , we say that I is a set-theoretic complete intersection. We discuss the relation between the arithmetical rank $\text{ara } I$ and the linear quotient index $q(I)$ of a matroidal ideal in the final section. The main result we obtain is as below.

Theorem 4.4. *Let I be a matroidal ideal of degree d of a polynomial ring $R = K[x_1, \dots, x_n]$ and $\text{supp}(I) = \{x_1, \dots, x_n\}$. Then $\text{ara } I = q(I) + 1$ if one of the following conditions holds:*

- (i) I is square-free Veronese;
- (ii) $I = J_1 J_2 \dots J_d$ such that each J_i is generated by h distinct variables;
- (iii) $d = 2$,

where h is the height of I .

As a consequence of the above theorem, we have the corollary.

Corollary 4.5. *Let $I \subseteq K[x_1, \dots, x_n]$ be a matroidal ideal such that $\text{supp}(I) = \{x_1, \dots, x_n\}$. Then I is Cohen-Macaulay if and only if it is a set-theoretic complete intersection.*

2 Linear quotients and matroidal ideals

Throughout, $R = K[x_1, \dots, x_n]$ is the polynomial ring in n variables over a field K . By Cohen-Macaulay for an ideal I , we mean that the quotient ring R/I is Cohen-Macaulay. For a monomial ideal I , we define the support of I to be the set $\text{supp}(I) = \bigcup_{u \in G(I)} \text{supp}(u)$. In this section, we discuss the linear quotient index $q(I)$ of a matroidal ideal. We first recall the following definition.

Definition 2.1. *We say that a monomial ideal $I \subseteq R$ has linear quotients if there is an ordering u_1, \dots, u_s of the monomials belonging to $G(I)$ with $\deg u_1 \leq \deg u_2 \leq \dots \leq \deg u_s$ such that, for each $2 \leq j \leq s$, the colon ideal $\langle u_1, u_2, \dots, u_{j-1} \rangle : u_j$ is generated by a subset of $\{x_1, \dots, x_n\}$.*

Let I be a monomial ideal with linear quotients with respect to the ordering $\{u_1, \dots, u_s\}$ of the monomials belonging to $G(I)$. We write $q_j(I)$ for the number of variables which is required to generate the colon ideal $\langle u_1, u_2, \dots, u_{j-1} \rangle : u_j$. Let $q(I) = \max\{q_j(I) \mid 2 \leq j \leq s\}$. From the fact [3, Corollary 1.6] that the length of the minimal free resolution of R/I over R is equal to $q(I) + 1$, we see that the index $q(I)$ is independent of the particular choice of the ordering of

the monomials which gives linear quotients. Moreover, by the Auslander-Buchsbaum formula, we have $\text{depth } R/I = n - q(I) - 1$. It then follows from the equality $\dim R/I = n - \text{ht}(I)$ that a monomial ideal I with linear quotients satisfies $\text{ht}(I) \leq q(I) + 1$ and is Cohen-Macaulay if and only if $\text{ht}(I) = q(I) + 1$. We summarize the above as the following proposition.

Proposition 2.2. *Let I be a monomial ideal of R with linear quotients. Then $\text{ht}(I) \leq q(I) + 1$; and I is Cohen-Macaulay if and only if $\text{ht}(I) = q(I) + 1$.*

As stated in the introduction, it is known that the matroidal ideals have linear quotients. Therefore all the above discussion applied to matroidal ideals. Next, we introduce two lemmas which are useful later. In the sequel, we say that I is a matroidal ideal of $K[x_1, \dots, x_n]$ if $\text{supp}(I) = \{x_1, \dots, x_n\}$.

Lemma 2.3. *Let $I \subseteq K[x_1, \dots, x_n]$ be a matroidal ideal of degree d ; and let x and y be variables in R such that $xy \nmid u$ for any $u \in G(I)$. If $xf \in G(I)$ for some monomial f of degree $d - 1$, then $yf \in G(I)$.*

Proof. Write $f = x_1 \cdots x_{d-1}$. The assertion is clear if $d = 2$ so we may assume that $d \geq 3$. Let $g = y_1 \cdots y_{d-1}$ be a monomial in R different from f such that $yg \in G(I)$ and $|\text{supp}(f) \cap \text{supp}(g)|$ is maximal. We may assume that $y_i = x_i$ for $i = 1, \dots, k$. Suppose that $k \leq d - 2$. Then by the definition of matroidal ideal there are integers $i, j \geq k + 1$ such that $\frac{yg}{y_j}x_i \in I$, which contradicts to the choice of g . Therefore, $f = g$ and the assertion holds. \square

Lemma 2.4. *Let $I \subseteq K[x_1, \dots, x_n]$ be a matroidal ideal of degree d . If there are $d + 1$ distinct variables $\{y, y_1, \dots, y_d\} \subseteq \{x_1, \dots, x_n\}$ such that $f = y_1 \cdots y_d \in I$, then there exists an integer i such that $\frac{f}{y_i} \in I$.*

Proof. The assertion is clear if d is small. We may assume that $d \geq 3$. Let $g = z_1 \cdots z_d$ be a monomial in I different from f such that $y \in \text{supp}(g)$ and $|\text{supp}(f) \cap \text{supp}(g)|$ is maximal. We may assume that $z_i = y_i$ for $i = 1, \dots, k$ and $z_d = y$. Suppose that $k \leq d - 2$. Then by the definition of matroidal ideal there are integers $i, j \geq k + 1$ such that $\frac{g}{z_j}x_i \in I$, which contradicts to the choice of g . Therefore, $k = d - 1$ and the assertion holds. \square

Theorem 2.5. *Let I be a matroidal ideal of degree d of the polynomial ring $R = K[x_1, \dots, x_n]$ with $\text{supp}(I) = \{x_1, \dots, x_n\}$. Then $q(I) = n - d$.*

Proof. Since I has linear quotients, there is an ordering u_1, \dots, u_s of the monomials belonging to $G(I)$ such that, for each $2 \leq j \leq s$, the colon ideal $\langle u_1, u_2, \dots, u_{j-1} \rangle : u_j$ is generated by a subset of $\{x_1, \dots, x_n\}$.

To show the assertion, it is enough to show that

$$\langle u_1, u_2, \dots, u_{j-1} \rangle : u_j \subseteq \{x_1, \dots, x_n\} - \text{supp}(u_j) \quad (1)$$

for each $2 \leq j \leq s$ and

$$\langle u_1, u_2, \dots, u_{s-1} \rangle : u_s = \{x_1, \dots, x_n\} - \text{supp}(u_s).$$

Write $u_j = x_{i_1} \cdots x_{i_d}$. If $x_{i_t} \in \langle u_1, u_2, \dots, u_{j-1} \rangle : u_j$ for some $t \leq d$, then $u_j \in \langle u_1, u_2, \dots, u_{j-1} \rangle$ as $\langle u_1, u_2, \dots, u_{j-1} \rangle$ is a square-free monomial ideal, a contradiction. Thus, (1) holds. By (1), to finish the proof, it suffices to show that $y \in \langle u_1, u_2, \dots, u_{s-1} \rangle : u_s$ if $y \notin \text{supp}(u_s)$. However, this follows by Lemma 2.4 with $u_s = y_1 \cdots y_d$. \square

Corollary 2.6. *Let I be a matroidal ideal of degree d of the polynomial ring $R = K[x_1, \dots, x_n]$. Then the projective dimension of the ideal I over R is $\text{pd}_R(I) = n - d$.*

Proof. Since the length of the minimal free resolution of R/I over R is $q(I) + 1$ (see [3, Corollary 1.6]), we obtain that $\text{pd}_R(I) = \text{pd}_R(R/I) - 1 = q(I) = n - d$ \square

3 Unmixed matroidal ideals

An ideal is *unmixed* if all its prime divisors are of the same height. This property is held by a Cohen-Macaulay ideal. In this section, we give characterizations of an unmixed matroidal ideal in terms of its height, degree, and the number of variables.

We first recall one special kind of matroidal ideals, the square-free Veronese ideals.

Example 3.1. *The square-free Veronese ideal of degree d in the variables $\{x_1, \dots, x_n\}$ is the ideal which is generated by all square-free monomials in $\{x_1, \dots, x_n\}$ of degree d . It is easy to see that the square-free Veronese ideals are matroidal and unmixed. In particular, from [2, Theorem 4.2] one see that the square-free Veronese ideals are the only case for Cohen-Macaulay matroidal ideals.*

We now give a characterization of matroidal ideal of degree 2.

Theorem 3.2. *Let $I \subseteq K[x_1, \dots, x_n]$ be a matroidal ideal of degree 2 with $\text{supp}(I) = \{x_1, \dots, x_n\}$. Then there are subsets S_1, \dots, S_m of $\{x_1, \dots, x_n\}$ such that the following hold:*

- (i) $m \geq 2$ and $|S_i| \geq 1$ for each i ;
- (ii) $S_i \cap S_j = \emptyset$ if $i \neq j$ and $\bigcup_{i=1}^m S_i = \{x_1, \dots, x_n\}$;
- (iii) if $x \in S_i$, $y \in S_j$ for $i \neq j$, then $xy \in G(I)$;
- (iv) if $x, y \in S_i$ for some i , then $xy \notin G(I)$.

Moreover, let P_i be the prime ideals generated by the set $\{x_1, \dots, x_n\} - S_i$ for each i . Then $P_1 \cap P_2 \cap \dots \cap P_m$ gives the primary decomposition of I .

Proof. Let $t - 1 = |\{x_i \mid i \neq 1, x_i x_1 \notin G(I)\}|$; then $1 \leq t \leq n - 1$. Without loss of generality, we may assume that $x_1 x_i \notin G(I)$ if $i = 2, \dots, t$ and $x_1 x_i \in G(I)$ if $i = t + 1, \dots, n$. We first show the following two statements:

- (a) $x_i x_j \in G(I)$ if $i \leq t$ and $j \geq t + 1$;
- (b) $x_i x_j \notin G(I)$ if $i, j \leq t$.

To show (a) holds, suppose on the contrary that $x_i x_j \notin G(I)$ for some $i \leq t$ and $j \geq t + 1$. Since $x_i \in \text{supp}(I)$, there is a variable x_k such that $x_i x_k \in G(I)$. Moreover, $x_1 x_j, x_i x_k \in G(I)$ and I is matroidal imply that either $x_i x_1$ or $x_i x_j$ is in $G(I)$, a contradiction. Thus (a) holds. For (b), suppose on the contrary that $x_i x_j \in G(I)$ for some $i, j \leq t$. Since $x_1 x_n, x_i x_j \in G(I)$, it follows from the exchange property of matroidal ideals that either $x_1 x_i$ or $x_1 x_j$ belongs to $G(I)$, a contradiction. Thus (b) holds.

Let $S_1 = \{x_1, \dots, x_t\}$. Observe that $\{x_i x_j \mid i \leq t, \text{ and } j \geq t + 1\}$ is a subset of $G(I)$. If $G(I) = \{x_i x_j \mid i \leq t, \text{ and } j \geq t + 1\}$ then by setting $S_2 = \{x_{t+1}, \dots, x_n\}$ and we are done.

Therefore, we may assume that there are $j, k \geq t + 1$ such that $x_j x_k \in G(I)$. Let I' be the monomial ideal in $K[x_{t+1}, \dots, x_n]$ generated by the set $G(I) - \{x_i x_j \mid i \leq t, \text{ and } j \geq t + 1\}$. Then $\text{supp}(I') \subseteq \{x_{t+1}, \dots, x_n\}$. In fact, $\text{supp}(I') = \{x_{t+1}, \dots, x_n\}$. For if not, then there is a variable x_l with $l \geq t + 1$ such that $x_l \notin \text{supp}(I')$. Since $x_l x_1, x_j x_k \in G(I)$ and I is matroidal, either $x_l x_j$ or $x_l x_k$ is in $G(I')$. Therefore, either $x_l x_j$ or $x_l x_k$ is in $G(I')$, a contradiction. We note that I' is a matroidal ideal of degree 2 of the polynomial ring $K[x_{t+1}, \dots, x_n]$. Thus, the assertion follows by induction.

Let P_i be the prime ideals generated by the set $\{x_1, \dots, x_n\} - S_i$. By the properties of S_i , it is easy to see that $P_i = I : y$ for every $y \in S_i$. Therefore each P_i is an associate prime ideal of I . Let $w \in P_1 \cap \dots \cap P_m$; then $w \cdot y \in I$ whence $y \in \bigcup_{i=1}^m S_i$. It follows that $(I : w) \supseteq \langle x_1, \dots, x_n \rangle$. Since I is reduced, I has no embedded prime ideals. Therefore $I : w = R$ and so that $w \in I$. Hence, $P_1 \cap \dots \cap P_m = I$; and this completes the proof. \square

From the above theorem we see that the S_i 's are uniquely determined. Moreover, if I is unmixed then we have that $|S_i| = |S_j| = n - ht(I)$ for all i, j . Therefore we have the following corollary.

Corollary 3.3. *Let $I \subseteq K[x_1, \dots, x_n]$ be an unmixed matroidal ideal of degree 2; then one has $\frac{n}{2} \leq ht(I) \leq n - 1$. In particular, $ht(I) = n - 1$ if and only if I is square-free Veronese; and $ht(I) = \frac{n}{2}$ if and only if n is even and $I = I_1 * I_2$ such that each I_i is generated by $\frac{n}{2}$ distinct variables.*

Proof. It is obvious that $ht(I) \leq n - 1$ and the equality holds when $m = n$ and $|S_i| = 1$ for all i , i.e., I is square-free Veronese. On the other hand, since $|S_1| + |S_2| = 2(n - ht(I)) \leq \sum_{i=1}^m |S_i| = n$, we obtain that $n \leq 2ht(I)$. This equality holds when $m = 2$ and in this case $I = I_1 * I_2$ such that each I_i is generated by $\frac{n}{2}$ distinct variables. \square

Here, we connect matroidal ideals with graphs. Observe that if $I \subseteq K[x_1, \dots, x_n]$ is a square-free monomial ideal of degree 2 then I defines a simple graph G with vertex set $\{x_1, \dots, x_n\}$ and edge set $\{x_i x_j \mid x_i x_j \in I\}$. If this is the case, we also say that I is the defining ideal of G . The following corollary is a consequence of Theorem 3.2.

Corollary 3.4. *Let I be a matroidal ideal of degree 2 of a polynomial ring $R = K[x_1, \dots, x_n]$. If I is the defining ideal of a simple graph G , then there are positive integers t_1, \dots, t_m such that $n = t_1 + \dots + t_m$ and $G = K_{t_1, t_2, \dots, t_m}$. In particular, if I is unmixed, then $G = K_{t, t, \dots, t}$.*

Example 3.5. *Let G be a graph defined by a matroidal ideal of degree 2 of the polynomial ring $R = K[x_1, \dots, x_6]$. If G is unmixed, then by Corollary 3.4, $G = K_6$ or $K_{3,3}$ or $K_{2,2,2}$.*

Next, we proceed to state and prove the main result in this section which gives a characterization of unmixed matroidal ideals of degree d in a polynomial ring $R = K[x_1, \dots, x_n]$.

Theorem 3.6. *Let I be a matroidal ideal of degree d of a polynomial ring $R = K[x_1, \dots, x_n]$, where $n \geq 2$. If I is unmixed, then $h + d - 1 \leq n \leq hd$, where h is the height of I . In particular, $n = h + d - 1$ if and only if I is square-free Veronese; and $n = hd$ if and only if $I = J_1 * J_2 * \dots * J_d$, where each J_i is generated by h distinct variables.*

Proof. Observe first that the assertion holds if $d = 1$ for that in this case $I = \langle x_1 \dots x_n \rangle$ and $n = h$. Therefore we assume that $d \geq 2$. We proceed the proof by induction on d . If $d = 2$,

then it is the content of Corollary 3.3. Thus we assume now that $d \geq 3$. For $i = 1, \dots, n$, let $S_i = \{\frac{u}{x_i} \mid u \in G(I), \text{ and } x_i \mid u\}$ and I_i be the ideal generated by S_i . Then I_i is a matroidal ideal of degree $d - 1$ with $\text{supp}(I_i) \subseteq \{x_1, \dots, \hat{x}_i, \dots, x_n\}$ and

$$I = \sum_{i=1}^n x_i I_i.$$

We will show that I_i is unmixed in the following. For this, we prove I_1 for example. Let P_1, \dots, P_r be the minimal primes of I that contain x_1 and Q_1, \dots, Q_s be the minimal primes of I that do not contain x_1 ; then

$$I = P_1 \cap \dots \cap P_r \cap Q_1 \cap \dots \cap Q_s$$

is a minimal primary decomposition of I . Therefore

$$x_1 I_1 \subseteq \langle x_1 \rangle \cap I \subseteq \langle x_1 \rangle \cap Q_1 \cap \dots \cap Q_s = \langle x_1 \rangle \cdot (Q_1 \cap \dots \cap Q_s) \subseteq x_1 I_1$$

as $\langle x_1 \rangle \cdot (Q_1 \cap \dots \cap Q_s) \subseteq I$. Hence, $I_1 = Q_1 \cap \dots \cap Q_s$ which is unmixed as $ht(Q_j) = h$ for all j .

To obtain the inequality $n \geq h + d - 1$, let $t_i = |\text{supp}(I_i)|$ for $i = 1, \dots, n$; then $t_i \leq n - 1$. Since I_i is an unmixed matroidal ideal of degree $d - 1$ and of height h , $t_i \geq h + (d - 1) - 1$ by induction. So $n \geq h + d - 1$ as $t_i \leq n - 1$. It is clear that if I is square-free Veronese then $n = h + d - 1$. Conversely, if $n = h + d - 1$, then $t_i = n - 1 = h + (d - 1) - 1$, so that I_i is square-free Veronese by induction, it follows that I is square-free Veronese as $I = \sum_{i=1}^n x_i I_i$.

To obtain $n \leq hd$, let $T = \{x_i \mid i \neq 1, x_1 x_i \mid u, \text{ for some } u \in G(I)\} \subseteq \text{supp}(I_1) \subseteq \{x_2, \dots, x_n\}$. For $i = 1, \dots, r$, choose $f_i \in Q_1 \cap \dots \cap Q_s - P_i$. Let $y \in \{x_2, \dots, x_n\} - T$; then $x_1 f_i \in I$, so that $y f_i \in I \subseteq P_i$ by Lemma 2.3, it follows that $y \in P_i$ for every i . Therefore $h = ht(P_i) \geq 1 + n - 1 - |T| = n - t$, where $t = |T|$. Now, I_1 is an unmixed matroidal ideal of degree $d - 1$ and of height h . By induction $h(d - 1) \geq |\text{supp}(I_1)| \geq t \geq n - h$. Therefore, $hd \geq n$. It is clear that if $I = J_1 * J_2 * \dots * J_d$ such that each J_i is generated by h distinct variables, then $n = hd$. Conversely, if $n = hd$, then P_i is generated by the set $\{x_1, \dots, x_n\} - T$, so that $r = 1$ and $I = P_1 * (Q_1 \cap \dots \cap Q_s)$. The assertion follows as by induction. \square

4 Arithmetical rank of a matroidal ideal

The goal of this section is to study the *arithmetical rank* of a matroidal ideal. For this we recall the definition of *arithmetical rank* as follows.

Let R be a Noetherian ring and I be an ideal of R . We say that the elements $x_1, \dots, x_m \in R$ generate I up to radical if $\sqrt{(x_1, \dots, x_m)} = \sqrt{I}$. The minimal number m with this property is called the *arithmetical rank* of I , denoted by $\text{ara } I$. If $\mu(I)$ is the minimal number of generators for I and $ht(I)$ is the height of I , then it is known that

$$ht(I) \leq \text{ara } I \leq \mu(I).$$

I is called *set-theoretic intersection* if $ht(I) = \text{ara } I$. The following results will be used later.

Lemma 4.1. [6] *Let P be a finite subset of a ring R . Let P_0, \dots, P_r be subsets of P such that*

(i) $\bigcup_{i=0}^r P_i = P$;

(ii) P_0 has exactly one element;

(iii) if p and p' are different elements of P_i ($0 < i \leq r$) there is an integer i' with $0 \leq i' < i$ and an element in $P_{i'}$ which divides pp' .

If $q_i = \sum_{p \in P_i} p$, then

$$\sqrt{P} = \sqrt{(q_0, \dots, q_r)}.$$

Lemma 4.2. Let I and J be two monomial ideals of $R = K[x_1, \dots, x_n]$ such that $\text{supp}(I) \cap \text{supp}(J) = \emptyset$. Suppose that $\text{ara } I = u$ and $\text{ara } J = v$. Then $\text{ara}(I * J) = u + v - 1$.

Proof. See [6, Theorem 1], for example. \square

Lemma 4.3. Let I be a matroidal ideal of degree d of a polynomial ring $R = K[x_1, \dots, x_n]$. If $\text{ara } I \leq q(I) + 1$, then $\text{ara } I = q(I) + 1$.

Proof. From [4], we know that for I , the following holds:

$$pd_R R/I \leq \text{ara } I.$$

However, by [3, Corollary 1.6]

$$pd_R R/I = q(I) + 1.$$

Thus, the assertion holds. \square

Theorem 4.4. Let I be a matroidal ideal of degree d of a polynomial ring $R = K[x_1, \dots, x_n]$ and $\text{supp}(I) = \{x_1, \dots, x_n\}$. Then $\text{ara } I = q(I) + 1$ if one of the following holds:

(i) I is square-free Veronese;

(ii) $I = J_1 J_2 \cdots J_d$ such that each J_i is generated by h distinct variables;

(iii) $d = 2$,

where h is the height of I .

Proof. By Lemma 4.3 and Theorem 2.5, it is enough to show that $\text{ara } I \leq n - d + 1$.

(i) Suppose that I is square-free Veronese. Let S_i be the set of all square-free monomials of degree d in variables x_1, x_2, \dots, x_i ; then $|S_d| = 1$ and $S_d \subset S_{d+1} \subset \cdots \subset S_n$. Let $P_0 = S_d$ and $P_i = S_{d+i} - S_{d+i-1}$ for $i = 1, \dots, n - d$; then it is easy to check that $P = \bigcup_{i=0}^{n-d} P_i$ satisfies the assumptions of Lemma 4.1 and $P = G(I)$. Thus, $\text{ara } I \leq n - d + 1$.

(ii) Since $\text{ara } J_i = h$, $\text{ara } I \leq hd - d + 1 = n - d + 1$ follows by Lemma 4.2.

(iii) By Theorem 3.2, we can divide the set $\{x_1, \dots, x_n\}$ into subsets:

$\{x_1, \dots, x_{t_1}\}, \{x_{t_1+1}, \dots, x_{t_1+t_2}\}, \dots, \{x_{t_1+\dots+t_{m-1}+1}, \dots, x_n\}$ such that $n = t_1 + \cdots + t_m$ and

$x_i x_j \in I$ if and only if $i \leq t_1 + \cdots + t_l < j$ for some positive integer l . We may further assume that $t_m \leq t_{m-1} \leq \cdots \leq t_1$ and arrange the generators of I as follows:

$$\begin{array}{ccccccc}
x_1 x_{t_1+1} & \cdots & \cdots & \cdots & \cdots & \cdots & x_1 x_n \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
x_{t_1} x_{t_1+1} & \cdots & \cdots & \cdots & \cdots & \cdots & x_{t_1} x_n \\
x_{t_1+1} x_{t_1+t_2+1} & \cdots & \cdots & \cdots & x_{t_1+1} x_n & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & & \\
x_{t_1+t_2} x_{t_1+t_2+1} & \cdots & \cdots & \cdots & x_{t_1+t_2} x_n & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & & \\
x_{n-t_m-t_{m-1}+1} x_{n-t_m+1} & \cdots & x_{n-t_m-t_{m-1}+1} x_n & & & & \\
\cdots & \cdots & \cdots & & & & \\
x_{n-t_m} x_{n-t_m+1} & \cdots & x_{n-t_m} x_n & & & &
\end{array}$$

From the above figure we can construct an $(t_1 + \cdots + t_{m-1}) \times (t_2 + \cdots + t_m)$ matrix $A = [y_{ij}]$ with entries in I as follows: For every positive integer $i \leq t_1 + \cdots + t_{m-1}$, there is a unique nonnegative integer $k \leq m-2$ such that $t_1 + \cdots + t_k + 1 \leq i \leq t_1 + \cdots + t_{k+1}$. Then $y_{ij} = x_i x_{t_1 + \cdots + t_{k+1} + j}$ if $1 \leq j \leq t_{k+2} + \cdots + t_m$ and $y_{ij} = 0$ otherwise. Observe that A has the following properties:

- (a) If $y_{ij} \in G(I)$, then $y_{ii'} \in G(I)$ whenever $i' \leq j$.
- (b) $y_{ij} = 0$, then $y_{ii'} = 0$ whenever $i' \geq j$.
- (c) $y_{ij} = 0$ whenever $i + j \geq n + 1$.
- (d) Every generator of $G(I)$ is an entry of A .

Now let $P_0 = \{x_1 x_{t_1+1}\}$ and $P_1 = \{x_1 x_{t_1+2}, x_2 x_{t_1+1}\}$. In general, for $0 \leq l < \infty$, let

$$P_l = \{y_{ij} \in G(I) \mid i + j = l + 2\}.$$

Then by (c), $P_l = \emptyset$ if $l \geq n-1$. Therefore by (d), $G(I) = \bigcup_{l=0}^{\infty} P_l = \bigcup_{l=0}^{n-2} P_l$ and $|P_0| = 1$. Thus, it remains to check that the assumption (iii) of Lemma 4.1 holds. To see this, let $y_{ij}, y_{i'j'} \in P_l$ for some $l \geq 1$. We may assume that $i < i'$. To finish the proof, we need to discuss the following two cases:

Case 1. x_i and $x_{i'}$ are independent, i.e., $x_i x_{i'} \notin G(I)$. In this case, let k be the integer such that $t_1 + \cdots + t_k + 1 \leq i < i' \leq t_1 + \cdots + t_{k+1}$; then $y_{ij} = x_i x_{t_1 + \cdots + t_{k+1} + l + 2 - i}$ and $y_{i'j'} = x_{i'} x_{t_1 + \cdots + t_{k+1} + l + 2 - i'}$. Since $l + 2 - i' < l + 2 - i$, we see that $j' < j$, it follows by (a) that $y_{ij'} \in G(I)$. Moreover, $y_{ij'} \in P_{l'}$ for some $l' < l$ and $y_{ij'}$ divides $y_{ij} \cdot y_{i'j'}$, the assertion follows.

Case 2. x_i and $x_{i'}$ are dependent, i.e., $x_i x_{i'} \in G(I)$. In this case, there are two integers $k < k'$ such that $t_1 + \cdots + t_k + 1 \leq i \leq t_1 + \cdots + t_{k+1}$ and $t_1 + \cdots + t_{k'} + 1 \leq i' \leq t_1 + \cdots + t_{k'+1}$. Since $t_1 + \cdots + t_{k+1} < i' < n$, $1 \leq i' - (t_1 + \cdots + t_{k+1}) \leq t_{k+2} + \cdots + t_m$. Thus

$$\begin{aligned}
x_i x_{i'} &= x_i x_{t_1 + \cdots + t_{k+1} + i' - (t_1 + \cdots + t_{k+1})} \\
&= y_{i, i' - (t_1 + \cdots + t_{k+1})} \\
&\in P_{l'}
\end{aligned}$$

where $l' = i + i' - (t_1 + \cdots + t_{k+1}) - 2$. Observe that $t_1 + \cdots + t_{k+1} \geq i$ and $i' + j' = l + 2$ implies $i' - 2 < l$. We get $l' < l$. Since $x_i x_{i'}$ divides $y_{ij} \cdot y_{i'j'}$, the assertion follows. \square

The following corollary is a direct consequence of Proposition 2.2 and the above theorem.

Corollary 4.5. *Let $I \subseteq K[x_1, \dots, x_n]$ be a matroidal ideal such that $\text{supp}(I) = \{x_1, \dots, x_n\}$. Then I is Cohen-Macaulay if and only if it is a set-theoretic complete intersection.*

In view of Theorem 4.4, we propose the following conjecture.

Conjecture: Let I be a matroidal ideal of degree d of a polynomial ring $R = K[x_1, \dots, x_n]$. Then $\text{ara } I = n - d + 1$.

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